Written Homework #6

1 $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix} \Rightarrow$ nullity $= 4 - \text{rank}(A) = 2 \Rightarrow$ Therefore A has 2 L.I. columns

checking that span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$ is L.I.:

$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & -3 & -2 & 0 \\ 1 & 7 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & -3 & -2 & 0 \\ 0 & -3 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & -3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{17}{3} & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ only vectors 1 & 2 are L.I., so the basis $=$ span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix} \right\}$

So a matrix A that contains 2 L.I. columns $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$ is:

$$\boxed{A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & -3 & 0 & 0 \\ 1 & 7 & 2 & 3 \end{bmatrix}}$$

2 This is not possible, because the range$(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x - y + 2z = 0 \right\}$ defines a plane in $\mathbb{R}^3$, which must be spanned by 2 L.I. vectors. A linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ must have 2 columns & 3 rows, meaning that all of its columns are L.I. Therefore, it must be 1-1, since rank$(T) =$ # of columns of T.

3 (a) yes, $\{a_1, a_3\}$ is a basis for col(A) because those are the only pivot columns and therefore the only linearly independent vectors

(b) To determine if $\{a_1 + a_2, a_3 + a_4\}$ is L.I.

$b_1 + b_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, $b_3 + b_4 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$

$c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 3 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ } yes these vectors are L.I., and therefore form a basis for col(A)

$-2 \cdot 5 = -10$

(c) $\{a_1 - \frac{1}{3} a_3, a_4\} \Rightarrow c_1 \begin{bmatrix} 10/3 \\ -2/3 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 10/3 & 5 & 0 \\ -2/3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3/2 & 0 \\ -2/3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc} 1 & 3/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ columns are L.D. and therefore do not form a basis for col(A)

(d) $\{a_1 + a_3, a_4\} \Rightarrow c_1 \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 5 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The columns are L.I. and therefore form a basis for col(A)

4 (a) $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, col$(A) = \mathbb{R}^2$ (2 LI columns), null$(A) =$ span$\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \Rightarrow A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$a + b + c = 0$, $d + e + f = 0 \Rightarrow \boxed{A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}}$

(b) cannot exist, because by rank-nullity theorem dim(col(A)) + dim(null(A)) = # of columns, and that is not satisfied when the dimension of the nullspace is 2.

(c) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, row$(A) =$ span$\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \Rightarrow c_1 \begin{bmatrix} a \\ b \end{bmatrix} + c_2 \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} c_1 a \\ c_1 b \end{bmatrix} + \begin{bmatrix} c_2 c \\ c_2 d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

null$(A) =$ span$\left(\begin{bmatrix} 3 \\ -3 \end{bmatrix}\right) \Rightarrow A \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 3a - 3b = 0$, $3c - 3d = 0 \Rightarrow a = b$, $c = d$

rows a, b & c, d need to be scalar multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $A = \begin{bmatrix} 1 & 1 \\ t & t \end{bmatrix}$, and

$A \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ t & t \end{bmatrix} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ $3a = 3b$ ✓, $3t = 3t \Rightarrow t$ can be any # $\Rightarrow \boxed{A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}$

4 (d) $A=\begin{bmatrix}a&b\\c&d\\e&f\end{bmatrix}$, row(A) = span$\left(\begin{bmatrix}1\\1\end{bmatrix}\right) \Rightarrow c_1\begin{bmatrix}a\\b\end{bmatrix}+c_2\begin{bmatrix}c\\d\end{bmatrix}+c_3\begin{bmatrix}e\\f\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$

$\Rightarrow \left[\begin{array}{c|c}c_1a+c_2c+c_3e & 1\\ c_1b+c_2d+c_3f & 1\end{array}\right]$, null(A) = span$\left(\begin{bmatrix}3\\-3\end{bmatrix}\right) \Rightarrow$ rank(A) = 1 L.I. column

$\hookrightarrow A\begin{bmatrix}3\\-3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow 3a=3b,\ a=b,\ c=d,\ e=f$

$\boxed{A=\begin{bmatrix}1&1\\1&1\\0&0\end{bmatrix}}$

(e) $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, row(A) = span$\left(\begin{bmatrix}1\\2\end{bmatrix}\right) \Rightarrow c_1\begin{bmatrix}a\\b\end{bmatrix}+c_2\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix} \Rightarrow \left[\begin{array}{c|c}c_1a+c_2c & 1\\ c_1b+c_2d & 2\end{array}\right]$

null(A) = Span$\left(\begin{bmatrix}3\\1\end{bmatrix}\right) \Rightarrow$ rank A = 1 L.I. column

A is of form: $\begin{bmatrix}a&2a\\c&2c\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \left.\begin{matrix}3a+2a=0\\3c+2c=0\end{matrix}\right\}$ $\boxed{A=\begin{bmatrix}0&0\\0&0\end{bmatrix}}$

(f) $A=\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}$, col(A) = span$\left(\begin{bmatrix}1\\2\end{bmatrix}\right) \Rightarrow c_1\begin{bmatrix}a\\d\end{bmatrix}+c_2\begin{bmatrix}b\\e\end{bmatrix}+c_3\begin{bmatrix}c\\f\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

$\Rightarrow A=\begin{bmatrix}a&b&c\\2a&2b&2c\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow a+b+c=0,\ 2a+2b+2c=0 \Rightarrow a=1, b=2, c=-3$, e

$\boxed{A=\begin{bmatrix}1&2&-3\\2&4&-6\end{bmatrix}}$

5 (a) $B=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$, null(B) = span$\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \Rightarrow$ # of columns = 3, rank = 2, nullity = 1

BA will have nullity 2 if it has a rank of 1

$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \longrightarrow$ $\boxed{A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \Rightarrow \text{nullity of BA} = 2,\ \text{nullity of A} = 2}$

Yes, you could find a matrix A with a different nullity as long as it makes the matrix $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$ after multiplying with BA, for example if $A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$ (A has nullity 1)

(b) In order for BA to have nullity 1, it must have 2 LI columns

So: $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} \Rightarrow$ B can just be multiplied by itself, so $\boxed{A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}.\ \text{Nullity(A)} = 1.}$

Yes you could find a matrix with a different nullity as long as it makes the matrix B again after multiplication, so e.g. A could be $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ which has nullity 0.

(c) BA can never have nullity 0 because matrix multiplication would create all 0's in the last row, meaning the lowest nullity you could get is 1.

Written Homework 6

(1) Find a 3×4 matrix A with nullity 2 and with column space

$$\operatorname{col}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\},$$

or explain why such a matrix cannot exist.

(2) Give an example of a linear transformation from $T : \mathbb{R}^2 \to \mathbb{R}^3$ with the following two properties:

(a) T is not one-to-one, and

(b)

$$\operatorname{range}(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \,:\, x - y + 2z = 0 \right\};$$

or explain why this is not possible. If you give an example, you must include an explanation for why your linear transformation has the desired properties.

(3) Consider the following row equivalent matrices:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -1 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix}$$

Let $\operatorname{col}(A)$ be the column space of A. Answer the following with reasons.

(a) Is $\{\mathbf{a}_1, \mathbf{a}_3\}$ a basis for $\operatorname{col}(A)$?

(b) Is $\{\mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_3 + \mathbf{a}_4\}$ a basis for $\operatorname{col}(A)$?

(c) Is $\{\mathbf{a}_1 - \frac{1}{3}\mathbf{a}_3, \mathbf{a}_4\}$ a basis for $\operatorname{col}(A)$?

(d) Is $\{\mathbf{a}_1 + \mathbf{a}_3, \mathbf{a}_4\}$ a basis for $\operatorname{col}(A)$?

(4) Find, if possible, an example of a matrix A such that:

(a) A is 2×3, $\operatorname{col}(A) = \mathbb{R}^2$, and $\operatorname{null}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$.

(b) A is 2×3, $\operatorname{col}(A) = \mathbb{R}^2$, and $\operatorname{null}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$.

(c) A is 2×2, $\operatorname{row}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$, and $\operatorname{null}(A) = \operatorname{span}\left(\begin{bmatrix} 3 \\ -3 \end{bmatrix} \right)$.

(d) A is 3×2, $\operatorname{row}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$, and $\operatorname{null}(A) = \operatorname{span}\left(\begin{bmatrix} 3 \\ -3 \end{bmatrix} \right)$.

(e) A is 2×2, $\operatorname{row}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$, and $\operatorname{null}(A) = \operatorname{span}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)$.

(f) A is 2×3, $\operatorname{col}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$, and $\operatorname{null}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$.

(5) Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Note that $\operatorname{null}(B) = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

(a) Find if possible, a 3×3 matrix A, where BA has nullity 2. If you find an example, what is the nullity of the matrix A that you found? Can you find an example with a different nullity?

(b) Find if possible, a 3×3 matrix A, where BA has nullity 1. If you find an example, what is the nullity of the matrix A that you found? Can you find an example with a different nullity?

(c) Find if possible, a 3×3 matrix A, where BA has nullity 0. If you find an example, what is the nullity of the matrix A that you found? Can you find an example with a different nullity?

(6) Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Note that $\text{null}(B) = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$.

(a) Find if possible, a 3×3 matrix A, where BA has nullity 2. If you find an example, what is the nullity of the matrix A that you found? Can you find an example with a different nullity?

(b) Find if possible, a 3×3 matrix A, where BA has nullity 1. If you find an example, what is the nullity of the matrix A that you found? Can you find an example with a different nullity?

(c) Find if possible, a 3×3 matrix A, where BA has nullity 0. If you find an example, what is the nullity of the matrix A that you found? Can you find an example with a different nullity?